Stochastic analysis of the poroelastic theory –
One-dimensional case

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ABSTRACT

Poroelastic theory is a conscientious theory that it considers coupled relationship between deformation and excess pore pressure of soil body under stress. Poroelastic theory recently has been applied to solve settlement problems due to pumping and groundwater level fluctuations triggered by earthquakes. We developed stochastic poroelastic theory using displacements and excess pore pressure as basic variables to understand the coupled phenomena between soil and water in heterogeneous medium. The first-order-second-moment (FOSM) method was applied and the hydraulic conductivity was chosen as the only random variable. Mean and perturbation equations were solved, respectively, to investigate the mean and variance behaviors. Monte Carlo simulation was used to validate the FOSM approach. The results of one-dimensional case shows that hydraulic conductivity and excess pore pressure are proportional in the upper part and inversely proportional in the lower part at early time with drained condition at upper boundary. Though the results of covariance functions between the two stochastic approaches are not exact the same in magnitude, they show similar patterns. The FOSM method takes much less cumbersome calculation and computer resources than the Monte Carlo simulation. Besides, the results of Monte Carlo simulation are affected by numerical errors and realization number of the generated random fields. FOSM approach is shown to an effective tool in estimating the mean and covariance of a soil-water system.

Key Words: poroelastic theory, stochastic approach, First-order-second-moment method, Monte Carlo simulation, hydraulic conductivity

1. Introduction

The poroelasticity theory has been widely used to investigate the interactions between excess pore water pressure and displacement. The theory assumes that the stress-strain relation is linear and medium is isotropic and homogeneous. Terzaghi[1] developed the one-dimensional consolidation theory but the coupled effect between fluid and solid were not considered in his
theory. Biot[2] first proposed a fully coupled theory to account such an effect. Several researchers followed Biot’s approach and extended the theory to use more common parameters in the governing equations[3][4][5] and Kumpel[6] unified the confusing parameters in the poroelasticity theory used in various fields.

Since the poroelasticity theory is originally developed under the assumption of homogeneous media, the governing equations can’t be used to describe the field conditions with heterogeneous characters. However, the properties of the porous medium in field are often anisotropic and heterogeneous. The uncertainty caused by the heterogeneity need to be considered. Although Monte Carlo simulation is a commonly used stochastic method, (Frias et al.[7] and Ferronato et al.[8]) its efficient and accuracy has been criticized. We adopt first-order-second-moment (FOSM) stochastic approach to solve the first two moments, which are obtained by only solving the moment equations. In this study, Biot’s one-dimensional analytical solution is used to validate the mean solution while Monte Carlo simulation is to validate the covariance results from FOSM approach.

2. Poroelastic theory

The governing equations of poroelasticity theory in three dimensions with displacements and exceed pore pressure as variables were proposed in 1941 by Biot, which can be written, using notation index types, as (Kumpel[6])

\[
\begin{align*}
G \frac{\partial^2 u_i}{\partial x_i \partial x_j} + \frac{G}{1-2\nu} \frac{\partial \Delta V}{\partial x_i} - \alpha \frac{\partial P}{\partial x_i} &= 0 \\
\alpha \frac{\partial \Delta V}{\partial t} + Q^{-1} \frac{\partial P}{\partial t} - \kappa \frac{\partial P}{\partial x_i \partial x_j} &= 0
\end{align*}
\]

(1)

where \( x_i (i = 1, 2, 3) \) are \( x, y, \) and \( z \) coordinates in Cartesian system, \( t \) is time, \( u_i \) is the displacement in \( i \) axis, \( \Delta V = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \) is volume dilatation (\( \varepsilon_{ii} = \partial u_i / \partial x_i \)), \( P \) is excess pore pressure of pore fluid, \( G \) is shear modulus, \( \nu \) is Poisson ratio for drained condition, \( \alpha \) is dimensionless coefficient of effective stress, \( Q^{-1} \) is the compressibility introduced by Biot, and \( \kappa \) is Darcy conductivity \([LT^{-1}]\) and related to hydraulic conductivity \( K[LT^{-1}] \) \( K = \gamma_w \kappa \) with unit weight of water \( \gamma_w \). The governing equations mentioned above include three unknown displacements \( u_i \) and one unknown excess pore pressure \( P \). Five parameters \( G, \nu, \alpha, Q^{-1}, \) and \( \kappa \) are required before the system can be solved. However, the parameters vary in the spatial domain for a heterogeneous porous media. The classical governing equations of poroelastic theory need to be modified to account such an effect.

Following Biot’s approach and assuming local homogeneity and isotropy, we derive the governing equations for heterogeneous media as
\[
\frac{\partial}{\partial x_j} \left[ \mu \frac{\partial u_j}{\partial x_j} \right] + \frac{\partial}{\partial x_j} \left[ \lambda \frac{\partial u_j}{\partial x_j} - \alpha P \right] = 0
\]

\[
\frac{\partial}{\partial t} \left[ \alpha \frac{\partial u_j}{\partial x_j} + Q^{-1} P \right] - \frac{\partial}{\partial x_j} \left[ \kappa \frac{\partial P}{\partial x_j} \right] = 0
\]

where \( \mu \) and \( \lambda \) are Lame’s constants. Equation (2) is the governing equations used to determine the displacements and excess pore pressure in heterogeneous porous media under external forces. The governing equation (1) can be viewed as a special case of equation (2) for a homogeneous case. For a bedding layer system, the parameters keep the same in horizontal direction and only vary vertically. The governing equations can be simplified for one dimension as

\[
\frac{\partial}{\partial z} \left[ a^{-1} \frac{\partial w}{\partial z} - \alpha P \right] = 0
\]

\[
\frac{\partial}{\partial t} \left[ \alpha \frac{\partial w}{\partial z} + Q^{-1} P \right] - \frac{\partial}{\partial z} \left[ \kappa \frac{\partial P}{\partial z} \right] = 0
\]

where \( w \) is the vertical displacement (\( w = u_z \)), \( z \) is the vertical direction (\( z = x_i \)), and \( a \) is called the final compressibility (Biot[2]) and \( a^{-1} = \lambda + 2\mu \).

### 3. Application to one-dimensional case

The one-dimensional case considers a cylindrical soil with no water flow and no expansion in latter boundary. The bottom is a no-flow and a no-displacement boundaries while water can drain freely and soil can deform arbitrarily at upper surface as shown in figure 1 (Biot[2]).

#### 3.1 Stochastic approach

The one-dimensional governing equations (3) have four parameters \( a, \alpha, Q, \) and \( \kappa \). Among the parameters, the Darcy conductivity \( \kappa \) has the largest variability as listed in the lower part of table 1 and they were calculated from the upper part of table 1 by using the unit weight of water \( \gamma_w = 9810 \text{N/m}^3 \) and water compressibility \( \beta_w = 4.4 \times 10^{-10} \text{m}^2/\text{N} \). For simplification we assume that soil grain is incompressible with respect to the matrix, then \( \alpha \) equal to 1. Accordingly, the most variability parameter \( \kappa \) was firstly set to be the only random variable while others keep constants. The variables \( w \) and \( P \) are therefore also random. The stochastic first-order-second-moment (FOSM) method was adopted and each random variables are separated to two parts by their mean and small perturbation items as \( \kappa = \langle \kappa \rangle + \kappa', \quad w = \langle w \rangle + w', \quad \) and \( P = \langle P \rangle + P' \). Then the governing equations can be separated into mean equations (4) and perturbation equations (5) as
Mean equations (4) can be solved directly with analytical or numerical approaches and perturbation equations (5) is used to calculate covariance functions.

Table 1. The variability of parameters and the values used in the model for sandy medium.

(Rearrange from Freeze and Cherry, 1979, Bowles, 1968, and Das, 1994, 1997)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Total range</th>
<th>Adopted value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus (E)</td>
<td>1.38×10^6 ~ 1.73×10^8</td>
<td>1.00×10^8</td>
<td>N/m^2</td>
</tr>
<tr>
<td>Poisson ratio (ν)</td>
<td>0.15 ~ 0.35</td>
<td>0.25</td>
<td>none</td>
</tr>
<tr>
<td>Porosity (n)</td>
<td>0.25 ~ 0.70</td>
<td>0.375</td>
<td>none</td>
</tr>
<tr>
<td>Hydraulic conductivity (K)</td>
<td>1.00×10^{-13} ~ 1.92×10^{-3}</td>
<td>1.00×10^{-5}</td>
<td>m/s</td>
</tr>
<tr>
<td>Final compressibility (a)</td>
<td>3.69×10^{-9} ~ 6.86×10^{-7}</td>
<td>8.33×10^{-9}</td>
<td>m^2/N</td>
</tr>
<tr>
<td>Coefficient of effective stress (α)</td>
<td>1</td>
<td>1</td>
<td>none</td>
</tr>
<tr>
<td>Biot’s compressibility (Q^{-1})</td>
<td>1.10×10^{-10} ~ 3.08×10^{-10}</td>
<td>1.65×10^{-10}</td>
<td>m^2/N</td>
</tr>
<tr>
<td>Darcy hydraulic conductivity (κ)</td>
<td>1.02×10^{-17} ~ 1.96×10^{-7}</td>
<td>1.02×10^{-9}</td>
<td>m^3/s/kg</td>
</tr>
</tbody>
</table>

Assume the Darcy hydraulic conductivity κ is a log-normal distributed stationary random variable. If the natural log conductivity \( Y = \ln(\kappa) \) is a Gaussian distribution the expectation \( m_Y = <Y> \) and covariance \( C_{Yx}(x,y) = <Y'(x),Y'(y)> \) fully describe the \( Y \) distribution. Then multiply \( Y'(z,t), \ w'(Z,\tau), \) and \( P'(Z,\tau) \) and take expected value we can find

\[
\begin{align*}
\begin{cases}
a^{-1} \frac{\partial^2 C_{w}(z,t;\tau)}{\partial z^2} - \alpha \frac{\partial C_{w}(z,t;\tau)}{\partial z} = 0 \\
\alpha \frac{\partial^2 C_{w}(z,t)}{\partial t \partial z} + Q^{-1} \frac{\partial C_{w}(z,t)}{\partial t} - \kappa \frac{\partial^2 C_{w}(z,t)}{\partial z^2} - \kappa_0 \frac{\partial}{\partial z} \left[ C_{\tau}(z,t;\tau) \frac{\partial}{\partial z} < P(z,t) > \right] = 0
\end{cases}
\end{align*}
\] (6)
where $\kappa_g$ is the geometric mean of Darcy hydraulic conductivity defined as $\kappa_g = \exp(m)$ and $C_{AB}(z,t;Z,\tau)$ is covariance function. Note that, the small perturbation items are with zero mean. If $C_{YY}$ is assume to be a stationary exponential covariance function, its covariance function can be written as

$$C_{YY}(z,Z) = \sigma_Y^2 \exp\left(-\frac{|z-Z|}{I_Y}\right)$$

where $\sigma_Y^2$ and $I_Y$ are the variance and integral scale of $Y$, respectively. The covariance functions of $C_{yw}(z,t;Z,\tau)$, $C_{yp}(z,t;Z,\tau)$, $C_{ww}(z,t;Z,\tau)$, $C_{pw}(z,t;Z,\tau)$, $C_{wp}(z,t;Z,\tau)$, and $C_{pp}(z,t;Z,\tau)$ can be solved by equation (6) to (8).
3.2 Numerical scheme

Implicit finite difference method is used to solve the mean and perturbation equations. The one-dimensional elements are shown in figure 2 with mesh-centered nodes. Central difference is used for spatial derivative and backward difference is used for time derivative. In order to verify the accuracy of the numerical solution, Biot’s analytical solution is adopted to validate the mean solution and the results of Monte Carlo simulations are compared to FOSM approach.

To reduce the effect of boundary condition, domain size of 15 integral scales is used. Each integral scale include five numerical grids. For a simple case we choose integral scale to be one meter and therefore, the total domain is 15 meter and each numerical grid is 0.2 meter. Eight thousand one-dimensional natural log-conductivity fields were generated with integral scale of one and variance of 0.1 using Sequential Gaussian Simulation approach in GSLIB (Deutsch and Journel[9]). The mean and variance of the natural log conductivity distributions varying with depth were shown in figure 3.

![Figure 3. Mean and variance of natural log-conductivity distribution for 8000 realizations.](image)

4. Results and Discussions

Figure 4(a) and figure 4(b) are the mean solutions of displacement and excess pore pressure compared between analytical and numerical solutions. The values between analytical and numerical solutions are very close and the results of implicit finite difference method are more close to the analytical solution than the statistic results of Monte Carlo simulation.

Figure 5 shows the auto-covariance solutions between displacement and excess pore water pressure solved by FOSM approach and figure 6 shows the statistic result calculated from 8000 realizations of Monte Carlo simulation. The patterns of the two stochastic methods are very similar. However, the exactly values are not the same. The discrepancy might be due to the natural log conductivity realizations generated by GSLIB shown in figure 3.

Figure 7 and figure 8 show the cross-covariance solutions between natural log conductivity and excess pore pressure and displacement solved by FOSM approach and Monte Carlo simulation,
respectively. The results fix the first item in the covariance function at depth of 7.5 m below ground surface. The results in these two approaches are similar in pattern but not in the same magnitude again.

Figure 4. Comparison between analytical solution, Monte Carlo simulation, and implicit finite difference solutions in mean equations of (a) displacement and (b) excess pore pressure.

Figure 5. The auto-covariance solutions solved by FOSM approach. Vertical line in the figures illustrates the middle depth at 7.5m beneath the surface.
Figure 6. The auto-covariance solutions solved by Monte Carlo simulation. Vertical line in the figures illustrates the middle depth at 7.5m beneath the surface.

Figure 7. The cross-covariance solutions solved by FOSM approach. Vertical line in the figures illustrates the middle depth at 7.5m beneath the surface and horizontal line indicates the zero covariance value.
5. Conclusion

Poroelastic theory is a conscientious theory that it considers coupled relationship between deformation and excess pore pressure of soil body under stress. We developed stochastic poroelastic theory using displacements and excess pore pressure as the basic variables to explore the coupled phenomena between soil and water in heterogeneous medium. The first-order-second-moment (FOSM) method was applied. The hydraulic conductivity was chosen as the random variable. Mean and perturbation equations were solved, respectively, to investigate the mean and variance behaviors. Monte Carlo simulations were used to compare with FOSM approach. Though the results of covariance functions between the two stochastic approaches are not exact the same in magnitude, they show similar patterns. The results of Monte Carlo simulation are affected by numerical errors and realization number of the generated random fields. FOSM approach is shown to an effective tool in estimating the mean and covariance of a soil-water system.

Reference


摘要

孔弾性理論被證實比傳統壓密理論更嚴謹，其考量土體受力之形變與超額孔隙水壓間之耦合關係，近年來被普遍的應用在探討抽水引起地層下陷的問題，與地震引致地下水位變化之機制等。為了了解土體與水體兩者在異質場中之耦合關係，本文使用最基本的位移與超額孔隙水壓為變數的孔弾性模式，導入一階二動差序率方法將水力傳導係數視為隨機變數，分別求取一維度案例之平均値方程式與共變差方程式，探討位移與超額孔隙水壓之平均行為與其變異性，並以蒙地卡羅法來與一階二動差法做比較。共變差函數之結果顯示，當時間較短且向上排水時，水力傳導係數與超額孔隙水壓在下半部呈現正比，上半部則呈現反比。即上半部之水力傳導係數相對較低時，下半部之超額孔隙水壓因向上排水受阻而相對較高；反之，上半部之水力傳導係數相對較高時，排水速度快而超額孔隙水壓相對較低。雖然兩種序率方法在共變差函數的結果有量上的差距，但其型態相當類似，其中一階二動差法在電腦的運算量與硬體的儲存量方面都遠低於蒙地卡羅法，且蒙地卡羅法可能受到隨機場之分佈不均與隨機場場數不確定性之影響，故以一階二動差法之結果較為良好而可信。

關鍵字：孔弾性理論、序率方法、一階二動差法、蒙地卡羅法、水力傳導係數