ELASTIC SOLUTIONS OF STRESSES FOR A TRANSVERSELY ISOTROPIC FULL-SPACE WITH INCLINED PLANES OF SYMMETRY SUBJECTED TO A POINT LOAD

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ABSTRACT

In this article, we present the analytical solutions for the stresses induced by three-dimensional point loads in a transversely isotropic full-space, which are not limited the planes of transversely isotropy parallel to the horizontal surface. In other words, the present solutions could calculate at any direction of stresses in a transversely isotropic full-space. The double Fourier transforms is utilized to obtain integral expressions of the Green’s stresses, then, the double inverse Fourier transforms and residue calculus carry out the contour integration involved. These solutions indicate that the stresses are affected by (1) the rotation of the planes of transverse isotropy ($\phi$), (2) the type and degree of material anisotropy ($E/E', \nu/\nu', G/G'$), (3) the geometric position ($r, \phi, \xi$), and (4) the loading types ($P_x, P_y, P_z$). Hence, in engineering practice, the dip at an angle of inclination must be considered when estimating the stresses in a transversely isotropic full-space subjected to applied loads.

Keywords: stresses, transversely isotropic full-space, double Fourier transforms, inverse Fourier transform, residue calculus

1.INTRODUCTION

Frequently in practice, the design of a foundation in soil/rock is governed by the stress criterion. This is particularly relevant to the cases where structures impose very large loads on the underlying soil/rock. However, generally, the magnitude and distribution of the stresses in soil/rock are predicted by using numerical/analytical solutions that model the constituted materials as a linearly
elastic, homogeneous and isotropic continuum. It is well recognized that for soils are deposited through a process of sedimentation over a long period of time, or rock masses cut by discontinuities, such as cleavages, foliations, stratifications, schistosities, joints, these solutions should account for anisotropy. Anisotropic soils/rocks are often modeled as transversely isotropic (cross anisotropic) materials from the standpoint of practical considerations in engineering. Nevertheless, the case where the discontinuities dip at an angle of inclination to the horizontal surface is commonly encountered, thus, such effect on the stresses are of interested us. In this work, elastic analytical solutions of stresses due to three-dimensional point loads in an inclined transversely isotropic full-space are derived.

In the middle of 1960, Doring [1, 2] theoretically considered the effect of inclined features by using a set of parallel joints, which reached the Mohr-Coulomb yield criterion. Grishin et al. [3] compared the theoretical values of stresses from an approximate analysis of an elastic model with experimental results found from a model of inclined gypsum layers. Gerrard and Harrison [4] calculated the total stresses and principal stresses within the orthogonal half-space by using finite element method. Their loading condition was that of uniform vertical pressure applied to the surface of the half-space, then, numerical results were computed for the angles of rotation of 0°, 30°, 45°, 60° and 90° between the direction of the maximum modulus and the horizontal. Gaziev and Erlikhman [5] investigated the distribution of vertical stress by model studies when the uniform loading was applied at an angle of 45° to stratification. Also, the patterns of stress distribution in the rock elements of the jointed medium depending on the angle of dip (0°, 30°, 45°, 60°, 90°) of the layers were presented. For the case of a line load inclined arbitrarily in an equivalent transversely isotropic half-space, Bray [6] decomposed the line load into two components that parallel and perpendicular to the planes of discontinuity, and concluded that only the radial stress existed in this medium. The properties of an equivalent transversely isotropic half-space were expressed as the elastic modulus $E$, Poisson’s ratio $\nu$, the normal stiffness $k_n$, the shear stiffness $k_s$, and the average spacing $S$ between discontinuities. Amadei and Pan [7] derived the closed-form solutions for gravitational stresses in the transversely isotropic, orthotropic, generally anisotropic rock masses with inclined strata under the assumption of no lateral horizontal strains. Recently, Ke and Lin [8] estimated the stress increments in an inclined transversely isotropic rock stratum subjected to surface strip loading by using commercial two-dimensional code, FLAC. Partial analysis results were compared with the rectangular loading solutions of Wang and Liao [9] by approaching the rectangle as either $l$ or $w$ into infinity. A series of charts were shown to study the stress increments in such a rock which was influenced by the type and degree of material anisotropy ($E/E’=1, 2, 3$), and the inclination angle with respect to the loading surface (0°, 30°, 45°, 60°, 90°). However, it was found that using two-dimensional FLAC code cannot realistically simulated the actual rock stratum.
Regarding the more correlative literature to our research, several investigators [10-20] have been presented the analytical solutions of stresses subjected to a point load in a transversely isotropic full-space, which the planes of transversely isotropy are parallel to the horizontal loading surface. So far to the best of our knowledge, no closed-form solutions of stresses where the planes of transversely isotropy dip at an angle of inclination for a transversely isotropic full-space subjected to three-dimensional point loads (Px, Py, Pz), as depicted in Figure 1, have been addressed. In deriving the formulation, we mainly follow the approaches proposed by Willis [12] for a transversely isotropic medium. That is, the triple Fourier transforms is utilized to obtain integral expressions of the Green’s stresses, then, the triple inverse Fourier transforms and residue calculus carry out the contour integration involved. Nevertheless, it should be noted that the major difference between Willis’s [12] and us is the chosen orthogonal vectors. In the former paper, there are two axes on the plane of transverse isotropy, and the third one is parallel to the rotation axis of elastic symmetry. It means a state of plane strain is utilized in his procedure. However, in the present article, the original co-ordinate system defines as x’, y’, z’, as shown in Figure 1. The new co-ordinate system x, y, z can be obtained from x’, y’, z’ by rotating an angle $\phi$ about the strike direction axis x(=x’). The degree of $\phi$ ranges from 0 to $2\pi$, hence, at any direction of the stresses in a transversely isotropic full-space can be calculated by the present solutions. These solutions indicate that the stresses are affected by (1). the rotation of the planes of transverse isotropy ($\phi$), (2). the type and degree of material anisotropy (E/E’, $\nu$/\nu’,G/G’), (3). the geometric position (r, $\phi$, $\xi$), and (4). the loading types (Px, Py, Pz).

2. DERIVING PROCEDURES

Figure 1 illustrates a transversely isotropic body, in which the z’ axis is the rotation axis of elastic symmetry, x’ and y’ axes in the plane of transverse isotropy. The deformability of a transversely isotropic material can be expressed as the following matrix form:

$$
\begin{bmatrix}
\sigma_{x'x'} & \tau_{x'y'} & \tau_{x'z'} \\
\tau_{y'x'} & \sigma_{y'y'} & \tau_{y'z'} \\
\tau_{z'x'} & \tau_{z'y'} & \sigma_{z'z'}
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{12} & C_{11} & C_{13} \\
C_{13} & C_{13} & C_{13}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{x'x'} \\
\varepsilon_{y'y'} \\
\varepsilon_{z'z'}
\end{bmatrix}
= \begin{bmatrix}
a_1 & a_1 - 2a_4 & a_3 - a_5 \\
a_1 - 2a_4 & a_1 & a_3 - a_5 \\
a_3 - a_5 & a_3 - a_5 & a_2
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_1 \\
a_2
\end{bmatrix}
= \begin{bmatrix}
\gamma_{x'x'} \\
\gamma_{y'y'} \\
\gamma_{z'z'}
\end{bmatrix}
= \begin{bmatrix}
\gamma_{x'x'} \\
\gamma_{y'y'} \\
\gamma_{z'z'}
\end{bmatrix}
$$

(1)

where $\sigma_{x'x'}$, $\sigma_{y'y'}$, $\sigma_{z'z'}$ are normal stresses; $\varepsilon_{x'x'}$, $\varepsilon_{y'y'}$, $\varepsilon_{z'z'}$ are normal strains;
τ_{xy}, τ_{xz}, τ_{yz} are shear stresses; γ_{xy}, γ_{xz}, γ_{yz} are shear strains; C_{11}, C_{12}, C_{13}, C_{33}, C_{44}, C_{66} are elastic moduli or elasticity constants. The elastic constant C_{12} is equal to C_{11} - 2C_{66}. Hence, only five elastic constants, i.e., C_{11}, C_{13}, C_{33}, C_{44}, C_{66} are independent for a transversely isotropic medium. The relation of C_{11}, C_{13}, C_{33}, C_{44}, C_{66} and a_1, a_2, a_3, a_4, a_5 can be expressed as follows:

\begin{align}
  a_1 &= C_{11}, \\
  a_2 &= C_{33}, \\
  a_3 &= (C_{44} + C_{13}), \\
  a_4 &= C_{66} = \frac{(C_{11} - C_{12})}{2}, \\
  a_5 &= C_{44}
\end{align}

Considering when a new co-ordinate system \(x', y', z'\) is obtained from the original system \(x, y, z\) by rotating an angle \(\phi\) about the strike direction axis \(x=x'\).

Hence, the new co-ordinate system of \([a]_{xyz}\) can be written as:

\[ [a]_{xyz} = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\
  a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\
  a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 \\
  a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & a_{55} & a_{56} \\
  0 & 0 & 0 & 0 & a_{65} & a_{66}
\end{bmatrix}. \]

The static point load (F_x, F_y, F_z), acting at the origin of the co-ordinate for a full-space can be expressed as the form of body forces:

\[ F_x = P_x \delta(x) \delta(y) \delta(z), \quad F_y = P_y \delta(x) \delta(y) \delta(z), \quad F_z = P_z \delta(x) \delta(y) \delta(z) \]

where \(\delta(\cdot)\) is the Dirac delta function.

Thus, the final expressions of stress solutions are:

\[ \sigma_{xx}(x,y,z) = -\frac{d}{d\omega} \left\{ \omega C_{a1}^1(\omega) \sigma_{xx}^1(\omega) \left( \frac{\Phi_1(\omega)}{\eta_1(\omega + \alpha_1)(\omega^2 - \beta_1^2)} \right) \right\}_{\omega \to \omega_1}, \]

\[ -\frac{d}{d\omega} \left\{ \omega C_{a2}^2(\omega) \sigma_{xx}^2(\omega) \left( \frac{\Phi_2(\omega)}{\eta_2(\omega + \alpha_2)(\omega^2 - \beta_2^2)} \right) \right\}_{\omega \to \omega_2}, \]

\[ -\frac{d}{d\omega} \left\{ \omega C_{a3}^3(\omega) \sigma_{xx}^3(\omega) \left( \frac{\Phi_3(\omega)}{\eta_3(\omega + \alpha_3)(\omega^2 - \beta_3^2)} \right) \right\}_{\omega \to \omega_3}, \]

\begin{align}
\end{align}
\( \sigma_{yy}(x, y, z) = -\frac{d}{d\omega} \left\{ \omega C_d^y(\omega) \sigma_{yy}^2(\omega) \left( \frac{\Phi_4(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_1 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^2(\omega) \sigma_{yy}^2(\omega) \left( \frac{\Phi_5(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_2 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^3(\omega) \sigma_{yy}^3(\omega) \left( \frac{\Phi_6(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_3 \)

\( \sigma_{zz}(x, y, z) = -\frac{d}{d\omega} \left\{ \omega C_d^y(\omega) \sigma_{zz}^2(\omega) \left( \frac{\Phi_4(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_1 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^2(\omega) \sigma_{zz}^2(\omega) \left( \frac{\Phi_5(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_2 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^3(\omega) \sigma_{zz}^3(\omega) \left( \frac{\Phi_6(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_3 \)

\( \tau_{zz}(x, y, z) = -\frac{d}{d\omega} \left\{ \omega C_d^y(\omega) \tau_{zz}^2(\omega) \left( \frac{\Phi_4(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_1 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^2(\omega) \tau_{zz}^2(\omega) \left( \frac{\Phi_5(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_2 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^3(\omega) \tau_{zz}^3(\omega) \left( \frac{\Phi_6(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_3 \)

\( \tau_{zz}(x, y, z) = -\frac{d}{d\omega} \left\{ \omega C_d^y(\omega) \tau_{zz}^2(\omega) \left( \frac{\Phi_4(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_1 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^2(\omega) \tau_{zz}^2(\omega) \left( \frac{\Phi_5(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_2 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^3(\omega) \tau_{zz}^3(\omega) \left( \frac{\Phi_6(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_3 \)

\( \tau_{xy}(x, y, z) = -\frac{d}{d\omega} \left\{ \omega C_d^y(\omega) \tau_{xy}^2(\omega) \left( \frac{\Phi_4(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_1 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^2(\omega) \tau_{xy}^2(\omega) \left( \frac{\Phi_5(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_2 \)

\( -\frac{d}{d\omega} \left\{ \omega C_d^3(\omega) \tau_{xy}^3(\omega) \left( \frac{\Phi_6(\omega)}{\eta_i(\omega + \alpha_i)(\omega^2 - \beta_i^2)} \right)^2 \right\} \omega \rightarrow \omega_3 \)

where

\( \sigma_{xx}^i = -i(\alpha a_{11} D_{11} + \beta a_{12} D_{21} + \beta a_{14} D_{14} + i(a_{14} D_{21} + a_{13} D_{31})) u_i \)
Elastic solutions of stress for a transversely isotropic full-space with inclined planes of symmetry subjected to a point load

\[
\begin{align*}
\sigma_{xy}^i &= -i(\alpha a_{12}D_{11}^i + \beta a_{22}D_{21}^i + \beta a_{32}D_{31}^i + i(a_{24}D_{21}^i + a_{34}D_{31}^i)u_i \\
\sigma_{zz}^i &= -i(\alpha a_{13}D_{11}^i + \beta a_{23}D_{21}^i + \beta a_{33}D_{31}^i + i(a_{34}D_{21}^i + a_{33}D_{31}^i)u_i \\
\tau_{yz}^i &= -i(\alpha a_{14}D_{11}^i + \beta a_{24}D_{21}^i + \beta a_{44}D_{31}^i + i(a_{44}D_{21}^i + a_{34}D_{31}^i)u_i \\
\tau_{zx}^i &= -i(a_{56}(\beta D_{11}^i + \alpha D_{21}^i) + a_{55}(\alpha D_{31}^i + iD_{11}^iu_i) \\
\tau_{xy}^i &= -i(a_{66}(\beta D_{11}^i + \alpha D_{21}^i) + a_{56}(\alpha D_{31}^i + iD_{11}^iu_i) \\
A_i &= \frac{a_i}{a_5} \\
A_{2,3} &= \frac{1}{2} \left[ \frac{a_2^2 + a_3^2 - a_5^2}{a_2 a_3} \pm \left( \frac{a_2^2 + a_3^2 - a_5^2}{a_2 a_3} \right)^2 - 4 \left( \frac{a_2}{a_5} \right)^2 \right]^{\frac{1}{2}} \\
\alpha_i^j &= -\eta_i^j + \sqrt{\eta_i^j - 4\eta_i^j \eta_3^j} \\
\beta_i^j &= -\eta_i^j - \sqrt{\eta_i^j - 4\eta_i^j \eta_3^j}. \\
\eta_1^j &= -(x + y - \sin 2\phi)(x - y + \sin 2\phi) \\
&\quad + 2A_j \sin \phi(2y \cos \phi + \sin \phi - \sin 3\phi - 2A_j \sin^3 \phi) + 2ix(y + (A_j - 1) \sin 2\phi) \\
\eta_2^j &= -1 - 2x^2 - 2y^2 + \cos 4\phi + 4y \sin 2\phi \quad - 2A_j(5 + 2 \cos 2\phi + \cos 4\phi + 2y \sin 2\phi) - 4A_j^2(3 + \cos 2\phi) \sin^2 \phi \\
\eta_3^j &= -(x + y - \sin 2\phi)(x - y + \sin 2\phi) \\
&\quad + 2A_j \sin \phi(2y \cos \phi + \sin \phi - \sin 3\phi - 2A_j \sin^3 \phi) - 2ix(y + (A_j - 1) \sin 2\phi) \\
The complete derivations for \( f_{ij} \) are listed in the Appendix C.
3. CONCLUSIONS

In this article, solutions are presented for the induced stresses by three-dimensional point loads in a transversely isotropic full-space, which the planes of transversely isotropy are arbitrarily oriented with respect to the horizontal surface. The double Fourier transforms in a Cartesian co-ordinate system are employed for deriving the analytical solutions. Besides, the double inverse Fourier transforms and residue calculus carry out the contour integration involved. These solutions are identical with the Liao and Wang’s solutions [20] if the full-space is homogeneous, linear elastic, and the planes of transverse isotropy are parallel to the horizontal surface. The exact solutions indicate that the stresses are affected by (1). the rotation of the planes of transverse isotropy ($\phi$), (2). the type and degree of material anisotropy ($E/E', \nu/\nu', G/G'$), (3). the geometric position ($r, \varphi, \xi$), and (4). the loading types ($P_x, P_y, P_z$).

REFERENCES

Elastic solutions of stress for a transversely isotropic full-space with inclined planes of symmetry subjected to a point load


摘要

本文係展示在横向等向性之無限空間其彈性對稱面不平行於水平受力面，當承受三向度點荷重作用下之應力解析解，換言之，本解可計算此無限空間內任意方向之應力值；在推導解的過程中，係利用二維 Fourier 變換、二維 Fourier 反變換、以及留數積分等方法求得；綜觀本解發現，其(1) 橫向等向面之旋轉角(φ)、(2) 異向性的種類與程度(E/E’, ν/ν’, G/G’)、(3)幾何位置(r, ϕ, ξ)、與(4)荷重的型式(Px, Py, Pz) 均會影響到本解，因此，於工程實務上，對於橫向等向性無限空間受外力作用後在估算應力量時宜應考慮傾角之影響。

關鍵字：應力、橫向等向性無限空間、二維 Fourier 變換、Fourier 反變換、留數積分